



6.1 Equations, linear equations, and systems of equations

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Introduction

- In this topic, we will
 - Discuss solutions to equations
 - Differentiate linear and non-linear equations
 - Define solution vectors and free variables
 - Look at solutions to non-linear and linear equations
 - Describe systems of equations
 - Consider a few examples of systems of non-linear equations
 - Describe a system of m linear equations in n unknowns





Equations

- An *equation* is any statement equating two mathematical expressions that have one or more *independent variables* or *unknowns*
 - For example, given three real numbers a , b and c , the *quadratic equation* equates a quadratic polynomial in one variable x to zero

$$ax^2 + bx + c = 0$$

- Given real numbers a through f , this equation may, if a solution exists, define a *conic section*:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$





Equations

- You could have an equation that involves various other functions, such as trigonometric, exponential, logarithmic and hyperbolic functions
 - The following are acceptable equations:

$$ax^2 + \sin(x) + c = 0$$

$$\sin(x) = 0$$

$$\ln(x) = 5$$

$$\sin(\ln(\operatorname{sech}(x) + 1) - 3y) = 1$$





Linear equations

- Some of the easiest equations involve only sums of products of unknowns with scalars, all equated to a target scalar:

$$6x_1 = 15$$

$$7x_1 - 8x_2 = 24$$

$$-4x_1 + 6x_2 - x_3 = -10$$

$$3x_1 + 5x_2 - 7x_3 + 8x_4 = 10$$

- Such equations are called *linear equations*





Solutions to equations

- If an equation has n unknowns, then a solution to that equation are any values these unknowns could take that make the equality true

- For example, given

$$4x_1 = 5$$

there is one solution: $x_1 = 1.25$

- For example, given the quadratic equation

$$2x_1^2 + 5x_1 + 2 = 0$$

has two solutions: $x_1 = -2$ and $x_1 = -0.5$

- For example, the equation

$$\sin(x_1) = 0$$

has as many solutions as there are integers: $x_1 = n\pi$ where n can be any integer





Solutions to equations

- In some cases, there may be many more than just a few solutions

- For example, given $4x_1 + 5x_2 = 12$

- then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.4 \end{pmatrix}$ is one solution,

- but also $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is another solution

- We can even *parameterize* the solutions:

- If we let c is a *free variable*, so if c is any real number

- then any pair $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c \\ 2.4 - 0.8c \end{pmatrix}$ is a solution





Solutions to equations

- In some cases, there may be many more than just a few solutions
 - For example, given

$$x_1^2 + x_2 = 0$$

then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is one solution,

and if c is a free variable with $c < 0$,

both $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{-c} \\ c \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\sqrt{-c} \\ c \end{pmatrix}$ are solutions





Comments for this course

- A few points:
 - If there are more than one variables or unknowns, we will describe a solution as a *solution vector*
 - In general, we will choose subscripted variables $x_1, \dots, x_n$
 - We could use different letters for free variables,

but this gets tedious

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c \\ 2.4 - 0.8c \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{c} \\ -c \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{c} \\ -c \end{pmatrix}$$

- Instead, if a vector is referred to as a *solution vector*, and it continues to have variables in the solution, those variables should be assumed to be free

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2.4 - 0.8x_1 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{-x_2} \\ x_2 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\sqrt{-x_2} \\ x_2 \end{pmatrix}_9$$





Linear equations

- Now suppose we have a linear equation in three unknowns:

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = b$$

- Note, we could write this as,

$$a_1 x_1 + a_2 x_2 + a_3 x_3 - b = 0$$

but having the target constant b on the right is standard

- Possible solution vectors include:

$$\mathbf{x} = \begin{pmatrix} \frac{b - a_3 x_3 - a_2 x_2}{a_1} \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \frac{b - a_3 x_3 - a_1 x_1}{a_2} \\ x_3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \frac{b - a_1 x_1 - a_2 x_2}{a_3} \end{pmatrix}$$





Non-linear equations

- Any equation in one or more unknowns that cannot be written as a linear equation is said to be a *non-linear equation*:

- For example,

$$x^2 + 3x + 2 = 0$$

$$xy = 5$$

$$\sin(x) = 7$$

$$e^x - 2 = 0$$

$$x + y + z^2 = 1$$

- Seldom, an equation may be a linear equation *in disguise*:

$$4 = \frac{6}{y} - 3\frac{x}{y} \quad 7x - 5y = \frac{1}{xy}$$

- This is, however, the exception and not the rule

- We will always write linear equations in the standard form





Non-linear equations

- With one variable x , a linear equation is easy to solve:

$$ax = b$$

has the solution $x = b/a$ assuming that $a \neq 0$

- With one variable, a non-linear equation may
 - Be difficult or even impossible to solve
 - Have any number of solutions include zero and infinitely many
 - Have solutions that are complex as well as real
- For example,

$$x^5 + 3x^4 + 2x^3 - 5x + 3 = 0$$

$$100 \sin(x) + x^2 = 100$$

$$e^x = 100 \quad \text{and} \quad e^x = 0$$





Systems of equations

- A collection of m equations in n unknowns is described as a *system of equations* if any solution vector must simultaneously satisfy all of the m equations
- If all m equations are linear equations,
we will describe it as a *system of linear equations*
 - We will refer to “systems of m linear equations in n unknowns”
- Any system of equations that has even one non-linear equation is referred to as a system of non-linear equations





Systems of non-linear equations

- For example, consider this system two non-linear equations:

$$x^2 - 2xy - 2 = 0$$

$$xy^2 + y - 1 = 0$$

- Let $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ represent a solution vector

- There is one real solution:

$$\mathbf{u} = \begin{pmatrix} 2 \\ 0.5 \end{pmatrix}$$

- It has two additional complex solutions:

$$\mathbf{u} = \begin{pmatrix} 1+j \\ j \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 1-j \\ -j \end{pmatrix}$$





Systems of non-linear equations

- Here is another system of two non-linear equations:

$$x^3 + x^2y + xy^2 + y^3 - 1 = 0$$

$$x^2 + xy - y^2 - 1 = 0$$

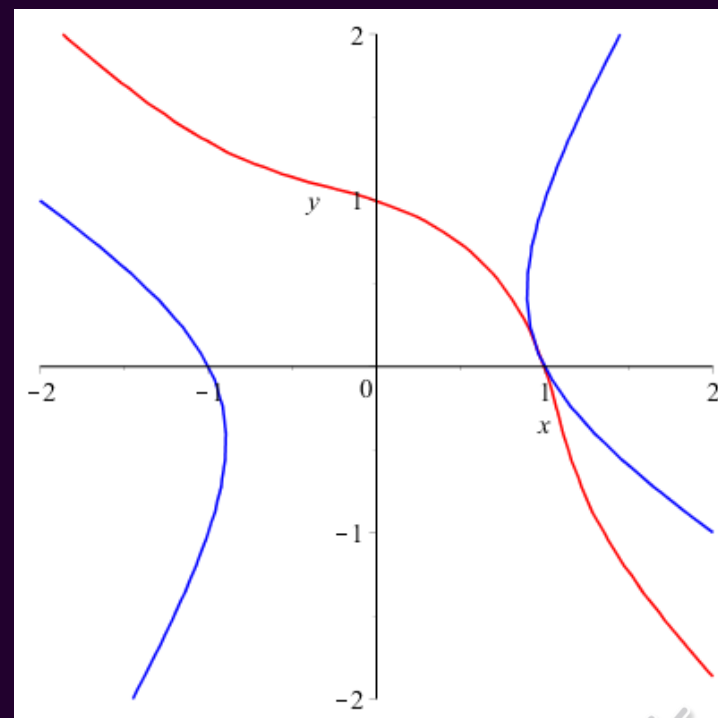
- It has one easy solution:

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- It has a second solution

$$\mathbf{u} = \begin{pmatrix} 0.9279 \\ 0.1878 \end{pmatrix}$$

but at best we can approximate it





Systems of linear equations

- The following is a system of linear equations in standard form:

$$3x - 4y + z = 16$$

$$2x + 3y - 4z = 1$$

$$x + 3z = 4$$

- Note: not all equations must contain all variables
 - Equation 3 does not depend on y

- If $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a solution vector, the unique solution is $\mathbf{u} = \begin{pmatrix} 3.25 \\ -1.5 \\ 0.25 \end{pmatrix}$





Systems of linear equations

- In general, we will have a system of m linear equations in n unknowns:

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n = b_2$$

$$a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + \cdots + a_{3,n}x_n = b_3$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + a_{m,3}x_3 + \cdots + a_{m,n}x_n = b_m$$

- We will refer to Eqn 1 through Eqn m

- A solution vector would be $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$





Summary

- Following this topic, you now
 - Understand the difference between linear and non-linear equations
 - Understand the difference between unknowns or variables and the idea of free variables in a solution vector
 - See that non-linear equations are more difficult to solve
 - Know the definition of a system of equations
 - Specifically, a system of linear equations
 - Understand the representation and terminology we will use with a system of m linear equations in n unknowns





References

- [1] https://en.wikipedia.org/wiki/System_of_linear_equations
- [2] https://en.wikipedia.org/wiki/Nonlinear_system





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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